Constrained Common Invariant Subspace and Its Application

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***Abstract*— The notion of constrained common invariant subspaces (CCISs) is proposed in this paper, as a generalization of the well-known invariant subspace, to study the structural properties of multiple matrices. Specifically, some necessary and sufficient conditions for the existence of a CCIS are established, which also provide a methodology to compute such a CCIS. Then, the properties of CCISs and their relation to common eigenvectors are revealed. It turns out that the existence of common eigenvectors leads to the existence of CCIS, but not vice versa, so the established CCIS can better reveal the structural properties of multiple matrices than common eigenvectors. The established CCIS is applied to the reducibility of Fornasini-Marchesini (F-M) state-space models, i.e., necessary and sufficient conditions and the related algorithm for reducibility of F-M models are developed. Finally, a gain-scheduled state-feedback control is proposed for a rational parameter system to further show the superiority of the established CCIS.**

***Index Terms*— Invariant subspace; order reduction; rational parameter system; state-feedback control.**

I. Introduction

THE invariant subspace (IS) of a single square matrix has been extensively studied and applied as the so-called geometric control theory for conventional linear time-invariant systems [1–3]. An essential application of the IS is to clarify the internal relationship between reducibility (or minimality) and controllability/observability in the framework of the geometric approach [4]. Take into account a linear system represented by

$$x(i+1) = Ax(i) + Bu(i),\ x \in \mathbb{R}^r,\ u \in \mathbb{R}^q \tag{1a}$$

$$y(i) = Cx(i) + Du(i),\ y \in \mathbb{R}^p. \tag{1b}$$

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It is well known that if there exists an IS $\mathcal{W}$ under the state matrix A satisfying $C\boldsymbol{w} = 0$ for every $\boldsymbol{w} \in \mathcal{W}$, then the linear system of (1) is not observable and can be exactly reduced [5]. Other concepts and their applications related to the IS, such as controlled IS and conditioned IS, have been discussed in several books (e.g. [4, 6]).

It is noted, however, multiple square matrices of equal size arise in many practical complex systems, e.g., n distinct state matrices $A_1, \ldots, A_n$ in linear parameter-varying systems [7] and Fornasini-Marchesini (F-M) state-space models [8–10]. The traditional IS cannot be applied to these multi-matrix cases. Therefore, in order to study the structural properties of multiple square matrices, the generalised common IS (CIS) has recently attracted much interest [11–14]. Pastuszak has revealed in [13] that CIS play an important role in quantifier elimination theory. Arroyo et al. have shown in [11] that CIS can overcome the challenges inherent in appropriately modelling graph differences and yet retain sufficient model simplicity to render estimation feasible. However, the existence conditions and computation of CIS remain a difficult problem [13, 14] because even generalising the results of IS to CIS is still a long-standing desire and challenge for researchers. Up to now, great efforts have been paid to attack this difficult problem. Shemesh in [15] has given a computable condition for the existence of a common eigenvector (CE), i.e., a special case of CIS with dimension one [16], of two matrices, which is generalized to a finite number of matrices in [17]. In [18], although the case of two matrices is taken into account, the authors have studied the CIS of dimensions higher than one, which, however, assumed that given matrices have pairwise different eigenvalues as pointed in [16]. Arapura and Peterson in [14] have given a general existence condition of CIS for an arbitrary finite number of matrices by converting the computation of CIS to find the projective variety of a series of symbolic equations by applying Gröbner basis technique, which, however, needs to know the dimension of CIS. Moreover, it is known that the Gröbner basis is time-consuming and difficult to achieve solutions for a large number of variables [19], making it difficult to apply to matrices with large sizes.

To avoid this difficulty, a possible way is to study the CIS with certain appropriate constraints. Along this end, we have recently made an attempt and established a notion of constrained CE (CCE) in [20], i.e., the one-dimensional CIS with a basis vector $\boldsymbol{w}$ satisfying $C\boldsymbol{w} = \mathbf{0}$. A necessary and sufficient condition for the existence of a CCE has also

been given in [20]. However, the existence condition and computation of the more general CIS with constraints and dimensions greater than one still remain unsolved.

Given the above background, this paper aims to establish a constrained CIS approach such that the structural properties of all the multiple square matrices can be explored. Specifically, a new notion called constrained CIS (CCIS) is given and necessary and sufficient conditions for the existence of a CCIS are established. Then, the properties of CCIS and its relation to CE are revealed. It turns out that the existence of a CE leads to the existence of a CCIS, but not vice versa, so the established CCIS can better reveal the structural properties of multiple matrices than the CE. Based on the advantages of CCIS to reveal the structural characteristics of multiple matrices, it has been applied to F-M models, where sufficient and necessary conditions and the related algorithms are developed for the reducibility of the F-M models. Moreover, a gain-scheduled state-feedback control is proposed for a rational parameter system, the numerical complexity of which can be greatly reduced by the established CCIS.

The organization of this paper is as follows. In Section II, the notion of CCIS is given, and then the necessary and sufficient conditions for the existence of CCIS are established. Section III reveals some properties of CCIS. Section IV applies the established CCIS into F-M models and presents sufficient and necessary reducibility conditions for the F-M models. Section V proposes an H_∞ gain-scheduled state-feedback controller synthesis for rational parameter systems. Conclusions are given in Section VI.

The following notations will be adopted in this paper. The r-dimensional real column and row vector spaces are denoted by $\mathbb{R}^r$ and $\mathbb{R}^{1\times r}$, respectively. We will denote the positive integers by $\mathbb{Z}_+$. For a number $n \in \mathbb{Z}_+$, let $\mathcal{N}$ represent the set $\{1,\ldots,n\}$. Let us denote by $\dim(\mathcal{M})$ the dimension of the space $\mathcal{M}$. A zero matrix with proper size and the specified size $p\times q$ will be correspondingly denoted by the symbols $\mathbf{0}$ and $\mathbf{0}_{p\times q}$. For a matrix $M \in \mathbb{R}^{p\times q}$, let $\mathrm{span}_{\mathrm{row}}(M)$ denote the space spanned by the row vectors of M; the kernel of M is $\mathrm{Ker}\,(M) = \{\boldsymbol{\eta} \in \mathbb{R}^q : M\boldsymbol{\eta} = \mathbf{0}\}$.

II. Constrained common invariant subspace

In this section, the notion of CCIS will be generalized from the notion of CIS [13] and CCE [20]. Then, the conditions for the existence of CCIS will be developed, which provide a technique to compute a CCIS and play an essential role in studying the structural properties of multiple square matrices.

Let $A_1,\ldots,A_n \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times q}$, $C \in \mathbb{R}^{p\times r}$, and $\mathcal{A} = \{A_1,\ldots,A_n\}$. We present the following notions.

Definition 1: A subspace $\mathcal{W}_{\mathrm{R}} \subseteq \mathbb{R}^r$ is called a right CIS of $A_1,\ldots,A_n$ if $A_1\boldsymbol{w} \in \mathcal{W}_{\mathrm{R}}$, …, $A_n\boldsymbol{w} \in \mathcal{W}_{\mathrm{R}}$ for all $\boldsymbol{w} \in \mathcal{W}_{\mathrm{R}}$ [13]. Dually, $\mathcal{W}_{\mathrm{L}} \subseteq \mathbb{R}^r$ is called a left CIS of $A_1,\ldots,A_n$ if $\boldsymbol{w}^\top A_1 \in \mathcal{W}_{\mathrm{L}}$, …, $\boldsymbol{w}^\top A_n \in \mathcal{W}_{\mathrm{L}}$ for all $\boldsymbol{w} \in \mathcal{W}_{\mathrm{L}}$.

Definition 2: A subspace $\mathcal{W}_{\mathrm{R}}$ is said to be a right CIS of $A_1,\ldots,A_n$ constrained by C if $C\boldsymbol{w} = \mathbf{0}$ for all $\boldsymbol{w}_{\mathrm{R}} \in \mathcal{W}$. Dually, $\mathcal{W}_{\mathrm{L}}$ is said to be a left CIS of $A_1,\ldots,A_n$ constrained by B if $\boldsymbol{w}^\top B = \mathbf{0}$ for all $\boldsymbol{w} \in \mathcal{W}_{\mathrm{L}}$. If it is not necessary to indicate the associate matrices explicitly, such a subspace will be just referred to as a right/left constrained CIS or CCIS. A right/left CCIS $\mathcal{W}$ is called trivial if $\dim(\mathcal{W}) = 0$; otherwise it is called nontrivial.

Definition 3: The matrix functions $M_k(\mathcal{A},C)$ and $\mathcal{M}_k(\mathcal{A},C)$ are recursively as defined by

$$M_k(\mathcal{A},C) \triangleq \begin{bmatrix} \mathcal{M}_{k-1}A_1 \\ \vdots \\ \mathcal{M}_{k-1}A_n \end{bmatrix} \in \mathbb{R}^{pn^{k-1}\times r}, \tag{2a}$$

$$\mathcal{M}_k(\mathcal{A},C) \triangleq \begin{bmatrix} M_1(\mathcal{A},C) \\ \vdots \\ M_k(\mathcal{A},C) \end{bmatrix} \in \mathbb{R}^{\frac{p\left(1-n^k\right)}{1-n}\times r}, \tag{2b}$$

with $\mathcal{M}_1(\mathcal{A},C) = M_1(\mathcal{A},C) = C$ and $k = 2,\ldots,\infty$.

The existence conditions of a right CCIS can be formulated to find a kernel of an infinite matrix as follows.

Theorem 1: There exists a nontrivial right CIS $\mathcal{W}$ of $A_1,\ldots,A_n \in \mathbb{R}^{r\times r}$ constrained by $C \in \mathbb{R}^{p\times r}$ if and only if $\mathcal{M}_\infty(\mathcal{A},C)$ is rank deficient. Moreover,

$$\mathcal{W} = \mathrm{Ker}\left(\mathcal{M}_\infty(\mathcal{A},C)\right), \tag{3}$$

with $\mathcal{M}_\infty(\mathcal{A},C) = \lim_{k\to\infty}\mathcal{M}_k(\mathcal{A},C)$.

Proof: We first prove the necessity and then sufficiency. Assume that $\mathcal{W}$ is a nontrivial right CCIS. Thus, $\dim(\mathcal{W}) \geq 1$ and for any $\boldsymbol{w} \in \mathcal{W}$ we have

$$A_i\boldsymbol{w} \in \mathcal{W}, \tag{4a}$$

$$C\boldsymbol{w} = 0. \tag{4b}$$

According to (4) and Definition 3, we derive

$$\mathcal{M}_\infty(\mathcal{A},C)\boldsymbol{w} = 0, \tag{5}$$

which gives the rank deficiency of $\mathcal{M}_\infty(\mathcal{A},C)$.

To show sufficiency, assume that $\mathcal{M}_\infty(\mathcal{A},C)$ is rank deficient. Let

$$\mathcal{W} \triangleq \mathrm{Ker}\left(\mathcal{M}_\infty(\mathcal{A},C)\right) \neq \emptyset. \tag{6}$$

Then, for any $\boldsymbol{w} \in \mathcal{W}$ we deduce

$$\mathcal{M}_\infty(\mathcal{A},C)\boldsymbol{w} = \mathbf{0}. \tag{7}$$

By the definition of $\mathcal{M}_k(\mathcal{A},C)$ and $M_k(\mathcal{A},C)$ in (2), we derive that

$$\mathcal{M}_k(\mathcal{A},C) = \begin{bmatrix} \mathcal{M}_{k-1}(\mathcal{A},C) \\ \mathcal{M}_{k-1}(\mathcal{A},C)A_1 \\ \vdots \\ \mathcal{M}_{k-1}(\mathcal{A},C)A_n \end{bmatrix}. \tag{8}$$

It follows from (7) and (8) that

$$\mathcal{M}_\infty(\mathcal{A},C)A_i\boldsymbol{w} = \mathbf{0}, i = 1,\ldots,n, \tag{9}$$

which indicates $A_i\boldsymbol{w} \in \mathcal{W}$. In view of $\mathcal{M}_\infty(\mathcal{A},C)$ containing $\mathcal{M}_1(\mathcal{A},C) = C$, we obtain

$$C\boldsymbol{w} = \mathcal{M}_1(\mathcal{A},C)\boldsymbol{w} = \mathbf{0}. \tag{10}$$

Therefore, by equation (10) and Definition 2 we conclude that $\mathcal{W}$ is a nontrivial right CCIS.

■

Lemma 1: If $\mathrm{span}_{\mathrm{row}}\mathcal{M}_{k+1}(\mathcal{A},C)=\mathrm{span}_{\mathrm{row}}\mathcal{M}_{k}(\mathcal{A},C)$, then $\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+2}(\mathcal{A},C)=\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(\mathcal{A},C)$.

Proof: Suppose that

$$\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+1}(\mathcal{A},C)=\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(\mathcal{A},C). \tag{11}$$

By Definition 3, we derive that

$$\begin{aligned}&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+1}(\mathcal{A},C)\\ =&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(\mathcal{A},C)+\mathrm{span}_{\mathrm{row}}\,M_{k+1}(\mathcal{A},C)\\ =&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(\mathcal{A},C)+\mathrm{span}_{\mathrm{row}}\mathcal{M}_{k}(\mathcal{A},C)A_1+\cdots\\ &+\mathcal{M}_{k}(\mathcal{A},C)A_n.\end{aligned} \tag{12}$$

It follows from (11) and (12) that

$$\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(A,C)A_i\subseteq\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(\mathcal{A},C), \tag{13}$$

which indicates that $\boldsymbol{w}^{\mathrm{T}}A_i\in\mathrm{span}_{\mathrm{row}}\,\mathcal{W}_{k}(\mathcal{A},C)$, $i=1,\ldots,n$, for all $\boldsymbol{w}\in\mathrm{span}_{\mathrm{row}}\,\mathcal{W}_{k}(\mathcal{A},C)=\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+1}(\mathcal{A},C)$. Then,

$$\begin{aligned}&\mathrm{span}_{\mathrm{row}}\,M_{k+2}(\mathcal{A},C)\\ =&\mathrm{span}_{\mathrm{row}}\begin{bmatrix}\mathcal{M}_{k+1}A_1\\ \vdots\\ \mathcal{M}_{k+1}A_n\end{bmatrix}\\ \subseteq&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+1}(\mathcal{A},C).\end{aligned} \tag{14}$$

According to Definition 3, we have

$$\begin{aligned}&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+2}(\mathcal{A},C)\\ =&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+1}(\mathcal{A},C)+\mathrm{span}_{\mathrm{row}}\,M_{k+2}(\mathcal{A},C).\end{aligned} \tag{15}$$

Combining (15) with (14) gives

$$\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+2}(\mathcal{A},C)=\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k+1}(\mathcal{A},C). \tag{16}$$

We conclude from (11) and (16) that $\mathrm{span}\,\mathcal{M}_{k+2}(\mathcal{A},C)=\mathrm{span}\,\mathcal{M}_{k}(\mathcal{A},C)$ and the proof is completed. ■

Lemma 2: $\mathrm{span}\,\mathcal{M}_{r}(\mathcal{A},C)=\mathrm{span}\,\mathcal{M}_{\infty}(\mathcal{A},C)$.

Proof: By Definition 3, we have

$$\begin{aligned}&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(\mathcal{A},C)\\ =&\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k-1}(A,C)+\mathrm{span}_{\mathrm{row}}\,M_{k-1}(\mathcal{A},C).\end{aligned} \tag{17}$$

In view of Lemma 1, we deduce that the dimension of the space $\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{r}(A,C)$ must be the maximum value of the dimension of the space of $\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{k}(\mathcal{A},C)$ for $k=1,\ldots,\infty$. Therefore, $\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{r}(\mathcal{A},C)=\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{r+1}(\mathcal{A},C)=\cdots=\mathrm{span}_{\mathrm{row}}\,\mathcal{M}_{\infty}(\mathcal{A},C)$ and then the proof is completed. ■

The existence condition of right CCIS given by Theorem 1 depends on an infinite matrix and can be converted equivalently into the following one, which is described by a finite-dimensional matrix.

Theorem 2: There exists a nontrivial right CCIS $\mathcal{W}$ of $A_1,\ldots,A_n\in\mathbb{R}^{r\times r}$ constrained by $C\in\mathbb{R}^{p\times r}$ if and only if $\mathcal{M}_r(\mathcal{A},C)$ is rank deficient. Moreover,

$$\mathcal{W}=\mathrm{Ker}\left(\mathcal{M}_r(\mathcal{A},C)\right). \tag{18}$$

Proof: It directly follows from the results of Lemma 2 and Theorem 1. ■

Algorithm 1: Computation of right CCIS

Input: $A_1,\ldots,A_n\in\mathbb{R}^{r\times r}$, $C\in\mathbb{R}^{p\times r}$.
Output: $\mathcal{W}$.

$\mathcal{M}_1\leftarrow C$;
for $k=2,\ldots,r$ **do**
  $M_k\leftarrow\begin{bmatrix}(\mathcal{M}_{k-1}A_1)^\top & \cdots & (\mathcal{M}_{k-1}A_n)^\top\end{bmatrix}^\top$;
  $M_k\leftarrow\begin{bmatrix}(\mathcal{M}_{k-1}A_1)^\top & \cdots & (\mathcal{M}_{k-1}A_n)^\top\end{bmatrix}^\top$;
  $\mathcal{M}_k\leftarrow\begin{bmatrix}\mathcal{M}_{k-1}^\top & M_k^\top\end{bmatrix}^\top$;
$\mathcal{W}\leftarrow\mathrm{Ker}\left(\mathcal{M}_r(\mathcal{A},C)\right)$.

Theorems 1 and 2 indicate an algorithm to derive a right CIS of $A_1,\ldots,A_n\in\mathbb{R}^{r\times r}$ constrained by $C\in\mathbb{R}^{p\times r}$, described by pseudocode in Algorithm 1.

Example 1: To demonstrate the specifics and effectiveness of Algorithm 1, take into account the following matrices

$$\begin{aligned}A_1=&\begin{bmatrix}0&0&1&1\\2&0&1&-1\\0&1&0&-1\\-1&2&3&2\end{bmatrix},\ A_2=\begin{bmatrix}2&1&0&0\\-2&2&1&1\\1&-3&-2&0\\1&2&1&1\end{bmatrix},\\ C=&\begin{bmatrix}1&1&1&-1\end{bmatrix}.\end{aligned} \tag{19}$$

By Algorithm 1, we obtain

$$\begin{aligned}&\mathcal{M}_4^\top=\begin{bmatrix}M_1^\top & M_2^\top & M_3^\top & M_4^\top\end{bmatrix}=\\ &\left[\begin{array}{c|cc|cccc|cccccccc}1&3&0&1&-4&4&2&-13&-8&0&6&8&-2&6&0\\1&-1&-2&-7&-2&-2&2&-9&6&-10&-2&6&6&-2&-4\\1&-1&-2&-7&-2&-2&2&-9&6&-10&-2&6&6&-2&-4\\-1&-3&0&-1&4&-4&-2&13&8&0&-6&-8&2&-6&0\end{array}\right],\end{aligned} \tag{20}$$

with

$$\begin{aligned}M_1^\top=&C^\top,\\ M_2^\top=&\begin{bmatrix}(CA_1)^\top & (CA_2)^\top\end{bmatrix},\\ M_3^\top=&\begin{bmatrix}(CA_1A_1)^\top & (CA_2A_1)^\top & (CA_1A_2)^\top & (CA_2A_2)^\top\end{bmatrix},\\ M_4^\top=&\Big[(CA_1A_1A_1)^\top\ (CA_2A_1A_1)^\top\ (CA_1A_2A_1)^\top\\ &(CA_2A_2A_1)^\top\ (CA_1A_1A_2)^\top\\ &(CA_2A_1A_2)^\top\ (CA_1A_2A_2)^\top\ (CA_2A_2A_2)^\top\Big].\end{aligned} \tag{21}$$

The kernel of $\mathcal{M}_4$ is

$$\mathcal{W}=\mathrm{Ker}\left(\mathcal{M}_4\right)=\mathrm{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2\}=\mathrm{span}\left\{\begin{bmatrix}0\\-1\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}, \tag{22}$$

which is a right CIS of A_1 and A_2 constrained by C.

Let $\mathcal{A}^\top\triangleq\{A_1^\top,\ldots,A_n^\top\}$. By the duality of left and right CCISs, the condition for the existence of the left CCIS can be given as follows.

Theorem 3: There exists a nontrivial left CIS $\mathcal{W}$ of $A_1,\ldots,A_n\in\mathbb{R}^{r\times r}$ constrained by $B\in\mathbb{R}^{r\times q}$ if and only if $\mathcal{M}_r(\mathcal{A}^\top,B^\top)$ is rank deficient. Moreover,

$$\mathcal{W}=\mathrm{Ker}\left(\mathcal{A}^\top,B^\top\right). \tag{23}$$

Since Theorem 3 can be proved similarly to Theorem 2, the details are omitted here for the sake of brevity.

III. Properties of CCIS

In the previous section, the existence conditions of the right and left CCISs have been given based on the ranks of the related matrices, separately. In this section, the existence of a right and left CCIS will be incorporated into one theorem. Then, the relationship between CCIS and CE will be revealed.

A. Existence conditions of CCISs

Definition 4: For a given $n \in \mathbb{Z}_+$, $\mathcal{F}_{\mathcal{N}}$ denotes the set consisting of sequences of elements in $\mathcal{N} \triangleq \{1, \ldots, n\}$. The elements of $\mathcal{F}_{\mathcal{N}}$ are also called as the strings or words [21]. Each $v \in \mathcal{F}_{\mathcal{N}}$ is in the form of

$$v = x_1 x_2 \cdots x_l$$

for some $x_k \in \mathcal{N}$, $k = 1, \ldots, l$, where x_k stands for the k-th letter of v, and $l = |v|$ is the length of v. Let ϵ denote the empty word, and $|\epsilon| = 0$. We will denote by $\mathcal{F}_{\mathcal{N}}^+$ the set of non-empty words.

Definition 5: For a finite collection of matrix $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{N}$, and an element $v = x_1 x_2 \cdots x_l \in \mathcal{F}_{\mathcal{N}}$, A_v is defined by

$$A_v = A_{x_l} A_{x_l - 1} \cdots A_{x_1} \in \mathbb{R}^{n\times n}, \tag{24}$$

where $A_v = I$ if $v = \epsilon$, i.e., $A_\epsilon = I$.

Definition 6 (Lexicographic Ordering): The lexicographic ordering $<$ on $\mathcal{F}_{\mathcal{N}}$ is defined as follows [22]. For any $v_1, v_2 \in \mathcal{F}_{\mathcal{N}}$ with $v_1 = \alpha_1\, \alpha_2 \cdots \alpha_{l_1}$ and $v_2 = \beta_1\, \beta_2 \cdots \beta_{l_2}$, $v_1 < v_2$ if either $|v_1| < |v_2|$, i.e., $l_1 < l_2$ or $0 < |v_1| = |v_2|$, $v_1 \neq v_2$ and for some $k \in \{1, \ldots, |v_1|\}$, $\alpha_k < \beta_k$ with the usual ordering of integers and $\alpha_i = \beta_i$ for $i = 1, \ldots, k-1$. It should be noted that $<$ is a complete ordering and

$$\mathcal{F}_{\mathcal{N}} = \{v_1,\ v_2, \ldots\}, \tag{25}$$

with $\epsilon = v_1 < v_2 < \cdots$.

Example 2: For $n = 2$, we have

$$\begin{aligned}\mathcal{F}_{\mathcal{N}} =& \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, \ldots\}\\ =& \{\ \epsilon,\ 1,\ 2, 11, 12, 21, 22, \ldots\}.\end{aligned} \tag{26}$$

Notation 1: Let $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times q}$, $C \in \mathbb{R}^{p\times r}$. Then, the matrix function $\mathcal{H}_k(\mathcal{A}, B, C)$ is defined by

$$\begin{aligned}&\mathcal{H}_k(\mathcal{A}, B, C)\\ =& \begin{bmatrix} CA_{v_1}B & CA_{v_2}B & \cdots & CA_{v_l}B\\ CA_{v_2}B & CA_{v_3}B & \cdots & CA_{v_{l+1}}B\\ \vdots & \ldots & \ddots & \vdots\\ CA_{v_l}B & CA_{v_{l+1}}B & \cdots & CA_{v_{2l-1}}B \end{bmatrix}\\ =&\mathcal{M}_k(\mathcal{A}, C)\mathcal{M}_k^\top(\mathcal{A}^\top, B^\top)\\ \in&\mathbb{R}^{\frac{p(1-n^k)}{1-n}\times\frac{q(1-n^k)}{1-n}},\end{aligned} \tag{27}$$

with $l = \frac{(1-n^k)}{1-n}$ and

$$\mathcal{M}_k(\mathcal{A}, C) \in \mathbb{R}^{\frac{p(1-n^k)}{1-n}\times r}, \tag{28a}$$

$$\mathcal{M}_k^\top(\mathcal{A}^\top, B^\top) \in \mathbb{R}^{r\times\frac{q(1-n^k)}{1-n}}. \tag{28b}$$

The existence condition of a right or left CCIS can be formulated into the following theorem by checking the rank of only one matrix.

Theorem 4: Let $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times q}$ and $C \in \mathbb{R}^{p\times r}$. Then, $A_1, \ldots, A_n$ share a nontrivial right CIS constrained by C or $A_1, \ldots, A_n$ share a nontrivial left CIS constrained by B if and only if

$$\operatorname{rank}\left(\mathcal{H}_r(\mathcal{A}, B, C)\right) < r. \tag{29}$$

Proof: We first prove the necessity and then sufficiency. Assume that $A_1, \ldots, A_n$ share a nontrivial right CIS constrained by C. According to Theorem 2, we have

$$\operatorname{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\right) < r. \tag{30}$$

By Sylvester inequality [23]:

$$\begin{aligned}\operatorname{rank}(X) + \operatorname{rank}(Y) - r \leq& \operatorname{rank}(XY)\\ \leq& \min\{\operatorname{rank}(X),\ \operatorname{rank}(Y)\},\end{aligned} \tag{31}$$

for $X \in \mathbb{R}^{m\times r}$ and $Y \in \mathbb{R}^{r\times n}$, we have

$$\begin{aligned}\operatorname{rank}\left(\mathcal{H}_r(\mathcal{A}, B, C)\right) =& \operatorname{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right)\\ \leq& \operatorname{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\right) < r.\end{aligned} \tag{32}$$

To show sufficiency, assume that the inequality of (29) holds true. By Sylvester inequality, we can get

$$\begin{aligned}&\operatorname{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\right) + \operatorname{rank}\left(\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right) - r\\ \leq& \operatorname{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right)\\ =& \operatorname{rank}\left(\mathcal{H}_r(\mathcal{A}, B, C)\right) < r,\end{aligned} \tag{33}$$

then

$$\operatorname{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\right) + \operatorname{rank}\left(\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right) < 2r. \tag{34}$$

It follows from (28) and (34) that

$$\operatorname{rank}\left(\mathcal{M}_r(\mathcal{A}, C)\right) < r \quad \text{or} \tag{35a}$$

$$\operatorname{rank}\left(\mathcal{M}_r^\top(\mathcal{A}^\top, B^\top)\right) < r, \tag{35b}$$

By Theorem 2, we conclude that $A_1, \ldots, A_n$ share a nontrivial right CIS constrained by C or $A_1, \ldots, A_n$ share a nontrivial left CIS constrained by B. ■

B. Relationship between CCIS and CE

To see more details on the effectiveness and novelty of the CCIS, we present some further comparisons to the representative CE approach.

A vector $\boldsymbol{w}$ is said to be a CE of A_i if and only if $\boldsymbol{w}$ is an eigenvector of every A_i i.e., there exists an eigenvalue λ_i of A_i satisfying $A_i\boldsymbol{w} = \lambda_i\boldsymbol{w}$ for every $i \in \mathcal{N} = \{1, \ldots, n\}$ [20, 24].

Theorem 5: If $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$ have a CE $\boldsymbol{w} \in \mathbb{C}^r$ satisfying

$$C\boldsymbol{w} = \boldsymbol{0}, \tag{36}$$

then $A_1, \ldots, A_n$ admit a CIS constrained by C; however, the reverse is not necessarily true.

Proof: The condition that $A_1, \ldots, A_n$ have a CE $\boldsymbol{w} \in \mathbb{C}^r$ satisfying (36) indicates that there exist eigenvalues $\lambda_i \in \mathbb{C}$ associated with the CE $\boldsymbol{w}$ such that

$$A_i\boldsymbol{w} = \lambda_i\boldsymbol{w},\ i = 1, \ldots, n. \tag{37}$$

Let $\boldsymbol{w}$ and $\lambda_i \in \mathbb{C}$ be expressed as

$$\lambda_i = \alpha_i + j\beta_i \tag{38a}$$
$$\boldsymbol{w} = \boldsymbol{\mu} + j\boldsymbol{\nu} \tag{38b}$$

with $\alpha_i, \beta_i \in \mathbb{R}$, $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathbb{R}^r$, and j being the imaginary unit. It follows from (37) and (38) that

$$\begin{aligned} A_i\boldsymbol{\mu} + jA_i\boldsymbol{\nu} =& A_i\boldsymbol{w} = \lambda_i\boldsymbol{w} = (\alpha_i + j\beta_i)(\boldsymbol{\mu} + j\boldsymbol{\nu}) \\ =& (\alpha_i\boldsymbol{\mu} - \beta_i\boldsymbol{\nu}) + j(\beta_i\boldsymbol{\mu} + \alpha_i\boldsymbol{\nu}) \end{aligned} \tag{39}$$

which gives that

$$A_i\boldsymbol{\mu} = \alpha_i\boldsymbol{\mu} - \beta_i\boldsymbol{\nu}, \tag{40a}$$
$$A_i\boldsymbol{\nu} = \beta_i\boldsymbol{\mu} + \alpha_i\boldsymbol{\nu}. \tag{40b}$$

From (36) and (38b), we obtain

$$C\boldsymbol{\mu} = C\boldsymbol{\nu} = \mathbf{0}. \tag{41}$$

We conclude from (39) and (41) that the space $\mathcal{W} = \mathrm{span}\{\boldsymbol{\mu}, \boldsymbol{\nu}\}$ is a CIS of $A_1, \ldots, A_n$ constrained by C.

However, the reverse is not true. Interesting is the fact that for the matrices A_1, A_2 and C given in (19), A_1 and A_2 have a CIS $\mathcal{W}$ of (22), whereas it can be checked there is no CE of A_1 and A_2 satisfying (36). ■

Remark 1: The concept of an IS of one matrix A is a similar one, and it is well-known that these are subspaces spanned by subsets of the eigenvectors of A [1]. However, as shown by the proof of Lemma 5, completely different from the IS, the CCIS of n matrices $A_1, \ldots, A_n$ are not necessarily spanned by a CE of $A_1, \ldots, A_n$. Therefore, the CCIS can better reveal the structural properties of multiple matrices than the CE.

IV. CCIS and Reducibility

This section is to apply the established CCIS to F-M models. Sufficient and necessary conditions and the related algorithm are developed for the reducibility of F-M models by utilizing CCIS.

A. CCIS of F-M model

The F-M model is described by [8, 25–28]

$$\begin{aligned} x(i_1+1, i_2+1, \ldots, i_n+1) =& A_1x(i_1, i_2+1, \ldots, i_n+1) \\ &+ \cdots + A_nx(i_1+1, \ldots, i_{n-1}+1, i_n) \\ &+ B_1u(i_1, i_2+1, \ldots, i_n+1) + \cdots \\ &+ B_nu(i_1+1, \ldots, i_{n-1}+1, i_n), \end{aligned} \tag{42a}$$
$$y(i_1, \ldots, i_n) = Cx(i_1, \ldots, i_n) + Du(i_1, \ldots, i_n), \tag{42b}$$

where $x(i_1, \ldots, i_n) \in \mathbb{R}^r$, $u(i_1, \ldots, i_n) \in \mathbb{R}^q$, and $y(i_1, \ldots, i_n) \in \mathbb{R}^p$ are the (local) state vector, the input vector and the output vector, respectively; r denotes the order; $A_1, \ldots, A_n \in \mathbb{R}^{r\times r}$, $B_1, \ldots, B_n \in \mathbb{R}^{r\times q}$, $C \in \mathbb{R}^{p\times r}$, $D \in \mathbb{R}^{p\times q}$. In the rest of the paper, we will denote by $(\mathcal{A}, \mathcal{B}, C, D; r)$ the F-M model of (42) with $\mathcal{A} \triangleq \{A_1, \ldots, A_n\}$ and $\mathcal{B} \triangleq \{B_1, \ldots, B_n\}$.

The transfer function matrix (TFM) of (42) is given by

$$H(z_1, \ldots, z_n) = C\left(I_r - \sum_{i=1}^{n} z_iA_i\right)^{-1}\left(\sum_{i=1}^{n} z_iB_i\right) + D, \tag{43}$$

where z_i represents the unit delay (or backward-shift) operator [8, 26].

From the F-M model of (42), we can just establish the left and right CCISs as follows.

Definition 7: For an F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$, a subspace $\mathcal{W}$ is said to be a right CCIS of $(\mathcal{A}, \mathcal{B}, C, D; r)$ if it is a right CCIS of $A_1, \ldots, A_n$ constrained by C. Dually, a subspace $\mathcal{V}$ is said to be a left CCIS of $(\mathcal{A}, \mathcal{B}, C, D; r)$ if it is a left CCIS of $A_1, \ldots, A_n$ constrained by B with

$$B = \begin{bmatrix} B_1 & \cdots & B_n \end{bmatrix}. \tag{44}$$

Remark 2: It should be noted that the IS is indeed very beneficial to the system identification, analysis and design, e.g., identification [2, 3], reducibility (or minimality) [4], controllability analysis [29], observability analysis [30], linear-quadratic mean field control [31], state feedback and observer design [32]. However, due to space limitations, the reducibility of systems will be investigated in particular based on CCIS in this paper. We then have the following result on F-M models.

B. Sufficient reducibility conditions for the F-M models

The reducibility to be taken into account for the F-M models is as follows [24].

Definition 8 (Reducibility of F-M models): For a given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$, if there is a new one $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ such that

$$\begin{aligned} & C\left(I_r - \sum_{i=1}^{n} z_kA_k\right)^{-1}\left(\sum_{k=1}^{n} z_kB_k\right) \\ =& \hat{C}\left(I_{\hat{r}} - \sum_{k=1}^{n} z_k\hat{A}_k\right)^{-1}\left(\sum_{k=1}^{n} z_k\hat{B}_k\right),\ \hat{r} < r, \end{aligned} \tag{45}$$

then the given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is reducible.

We want to remark that the inherently complex structural properties of the F-M models, such as those involving n different variable directions and corresponding n distinct state matrices $A_1, \ldots, A_n$, make it extremely difficult to derive minimal state-space models. Therefore, generalizing the well-known traditional IS theory to the situation with multiple matrices is a long-standing goal and challenge for many researchers. To this end, motivated by the IS for the traditional case a natural way is to directly explore the relationship between the IS and the reducibility of an F-M model. Then, a sufficient condition for the reducibility of an F-M model can be given by utilizing the proposed CCIS as follows.

Theorem 6: An F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is reducible if it admits a nontrivial right CCIS.

Proof: Assume that $A_1, \ldots, A_n$ share a nontrivial right CCIS $\mathcal{W}$ with a basis $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\}$ with $\tilde{r} \geq 1$. Choose

$\hat{r} \triangleq r - \tilde{r}$ vectors $\boldsymbol{w}_{\tilde{r}+1}, \ldots, \boldsymbol{w}_r$ to construct a nonsingular matrix

$$\begin{aligned} T &\triangleq \left[\begin{array}{ccc|ccc} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_{\tilde{r}} & \boldsymbol{w}_{\tilde{r}+1} & \cdots & \boldsymbol{w}_r \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} T_1 & T_2 \end{array}\right] \end{aligned} \tag{46}$$

and define

$$L \triangleq \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \triangleq T^{-1}, \tag{47}$$

with $L_1 \in \mathbb{R}^{\tilde{r}\times r}$ and $L_2 \in \mathbb{R}^{(r-\tilde{r})\times r}$.

We derive from the CIS of A_i, $i = 1, \ldots, n$, that $A_i\boldsymbol{w}_k$, $k = 1, \ldots, \tilde{r}$, can be linearly represented by the $\tilde{r}$ vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}$, i.e., there is a matrix $A_{i1} \in \mathbb{R}^{\tilde{r}\times\tilde{r}}$ such that

$$\left[\begin{array}{ccc} A_i\boldsymbol{w}_1 & \cdots & A_i\boldsymbol{w}_{\tilde{r}} \end{array}\right] = \left[\begin{array}{ccc} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_{\tilde{r}} \end{array}\right] A_{i1}. \tag{48}$$

Since $\boldsymbol{w}_k \in \mathbb{R}^n$ with $k \in \{\tilde{r}+1, \ldots, n\}$, $A_i\boldsymbol{w}_k \in \mathbb{R}^n$ must be linear represented by the n independent vectors $\boldsymbol{w}_1, \ldots \boldsymbol{w}_r \in \mathbb{R}^n$, i.e., there are matrices $A_{i2} \in \mathbb{R}^{\tilde{r}\times(r-\tilde{r})}$ and $A_{i4} \in \mathbb{R}^{(r-\tilde{r})\times(r-\tilde{r})}$ such that

$$\begin{aligned} &\left[\begin{array}{ccc} A_i\boldsymbol{w}_{\tilde{r}+1} & \cdots & A_i\boldsymbol{w}_r \end{array}\right] \\ =&\left[\begin{array}{ccc|ccc} A_i\boldsymbol{w}_1 & \cdots & A_i\boldsymbol{w}_{\tilde{r}} & A_i\boldsymbol{w}_{\tilde{r}+1} & \cdots & A_i\boldsymbol{w}_r \end{array}\right]\begin{bmatrix} A_{i2} \\ A_{i4} \end{bmatrix}. \end{aligned} \tag{49}$$

It follows form (46), (48) and (49) that

$$\begin{aligned} A_iT =& \left[\begin{array}{ccc|ccc} A_i\boldsymbol{w}_1 & \cdots & A_i\boldsymbol{w}_{\tilde{r}} & A_i\boldsymbol{w}_{\tilde{r}+1} & \cdots & A_i\boldsymbol{w}_r \end{array}\right] \\ =& \left[\begin{array}{ccc|ccc} \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_{\tilde{r}} & \boldsymbol{w}_{\tilde{r}+1} & \cdots & \boldsymbol{w}_r \end{array}\right] \begin{bmatrix} A_{i1} & A_{i,2} \\ \mathbf{0} & A_{i4} \end{bmatrix} \\ =& T\left[\begin{array}{c|c} A_{i1} & A_{i2} \\ \hline \mathbf{0} & A_{i4} \end{array}\right]. \end{aligned} \tag{50}$$

Pre-multiplying both sides of the above equation by $L = T^{-1}$ gives

$$LA_iT = \begin{bmatrix} A_{i1} & A_{i2} \\ \mathbf{0} & A_{i4} \end{bmatrix}. \tag{51}$$

By (46), (47) and (51), we obtain

$$A_{i1} = L_1A_iT_1,\ A_{i2} = L_1A_iT_2,\ A_{i4} = L_2A_iT_2. \tag{52}$$

According to that $\mathcal{W}$ is a CIS constrained by C, we derive

$$CT_1 = \left[\begin{array}{ccc} C\boldsymbol{w}_1 & \cdots & C\boldsymbol{w}_{\tilde{r}} \end{array}\right] \triangleq \left[\begin{array}{ccc} \mathbf{0} & \cdots & \mathbf{0} \end{array}\right] = \mathbf{0}, \tag{53}$$

and then

$$\begin{aligned} CT =& \left[\begin{array}{cc} CT_1 & CT_2 \end{array}\right] = \left[\begin{array}{cc} \mathbf{0} & CT_2 \end{array}\right] \\ \triangleq& \left[\begin{array}{cc} \mathbf{0} & C_{12} \end{array}\right]. \end{aligned} \tag{54}$$

It follows from (47) that

$$LB_i = \begin{bmatrix} L_1B_i \\ L_2B_i \end{bmatrix} \triangleq \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}. \tag{55}$$

On account of (51), (54), (55), we derive

$$\begin{aligned} &C\left(I_r - \sum_{i=1}^n z_iA_i\right)^{-1}\left(\sum_{i=1}^n z_iB_i\right) \\ =&CTT^{-1}\left(I_r - \sum_{i=1}^n z_iA_i\right)^{-1}L^{-1}L\left(\sum_{i=1}^n z_iB_i\right) \\ =&CT\left(I_r - \sum_{i=1}^n z_iLA_iT\right)^{-1}\left(\sum_{i=1}^n z_iLB_i\right) \\ =&\left[\mathbf{0}\ C_2\right]\left(I_r - \sum_{i=1}^n z_i\begin{bmatrix} A_{i1} & A_{i2} \\ \mathbf{0} & A_{i4} \end{bmatrix}\right)^{-1}\left(\sum_{i=1}^n z_i\begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}\right) \\ =&C_2\left(I_{\hat{r}} - \sum_{i=1}^n z_iA_{14}\right)^{-1}\left(\sum_{i=1}^n z_iB_{i2}\right), \end{aligned} \tag{56}$$

which indicates that a new F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ with lower order $\hat{r} = r - \tilde{r} < r$ is derived with

$$\begin{aligned} \hat{A}_i =& A_{i4} = L_2A_iT_2,\ \hat{B}_i = B_{i2} = L_2B_i, \\ \hat{C} =& C_2 = CT_2. \end{aligned} \tag{57}$$

■

Remark 3: It should be noted that the existence of a nontrivial right CCIS $\mathcal{W}$ of an F-M model indicates $\tilde{r} = \dim(\mathcal{W}) \geq 1$. By the proof of Theorem 6, the order of the new F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ is $\hat{r} = r - \tilde{r}$, which must be less than the order r of the given F-M model.

Based on the proof of Theorem 6, we give a reduction algorithm for the F-M model as shown in Algorithm 2.

Algorithm 2: Reducing F-M model using right CCIS

Input: $(\mathcal{A}, \mathcal{B}, C, D; r)$.
Output: $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$.

1. $\mathcal{W} = \mathrm{span}\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{\tilde{r}}\} \leftarrow$ Algorithm 1;
2. Construct a matrix T in the form of (46) and set matrix $L = T^{-1}$;
3. Determine T_2 from the matrix T of (46);
4. Obtain L_2 from the matrix L of (47);
5. Derive a new F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ by (57).

Example 3: To demonstrate the details and effectiveness of Algorithm 2, consider the F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ where $\mathcal{A} = \{A_1, A_2\}$ and C are given in (19), and

$$B_1 = \begin{bmatrix} 2 \\ -2 \\ 3 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ -2 \\ 4 \\ -1 \end{bmatrix},\ D = 0. \tag{58}$$

A CIS of A_1 and A_2 constrained by C has been given in (22). Then, we can construct a non-singular matrix as follows

$$T = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} T_1 & T_2 \end{array}\right], \tag{59}$$

and set

$$L = T^{-1} = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] \triangleq \left[\begin{array}{c} L_1 \\ \hline L_2 \end{array}\right]. \tag{60}$$

According to (47), a new 2-D F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ with lower order is determined by

$$\begin{aligned} &\hat{A}_1 = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \ \hat{A}_2 = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \\ &\hat{B}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \hat{B}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \\ &\hat{C} = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}, \ D = 0, \\ &\hat{r} = 4 - 2 = 2. \end{aligned} \tag{61}$$

The reducibility condition based on the right CCIS of the F-M model has been presented. In a similar manner, the reducibility condition can be drawn based on the left CCIS as well.

Theorem 7: An F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is reducible if it admits a nontrivial left CCIS.

Proof: Since the proof of Theorem 7 can be proved similarly as that of Theorem 6, we give only the main ideas of the proof.

Assume that $A_1, \ldots, A_n$ share a nontrivial left CCIS $\mathcal{V}$ with a basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\tilde{r}}\}$ with $\tilde{r} \geq 1$. Choose $\hat{r} \triangleq r - \tilde{r}$ vectors $\boldsymbol{v}_{\tilde{r}+1}, \ldots, \boldsymbol{v}_r$ to construct a nonsingular matrix

$$\begin{aligned} \hat{T}^\top &\triangleq \left[\begin{array}{ccc|ccc} \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{\tilde{r}} & \boldsymbol{v}_{\tilde{r}+1} & \cdots & \boldsymbol{v}_r \end{array}\right] \\ &\triangleq \left[\begin{array}{c|c} \hat{T}_1^\top & \hat{T}_2^\top \end{array}\right] \end{aligned} \tag{62}$$

and define

$$R \triangleq \begin{bmatrix} R_1 & R_2 \end{bmatrix} \triangleq \hat{T}^{-1}, \tag{63}$$

with $\hat{T}_1 \in \mathbb{R}^{\tilde{r} \times r}$, $\hat{T}_2 \in \mathbb{R}^{(r-\tilde{r}) \times r}$, $R_1 \in \mathbb{R}^{r \times \tilde{r}}$ and $R_2 \in \mathbb{R}^{r \times (r-\tilde{r})}$. It can be verified that the given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ can be exactly reduced to the new one $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, D; \hat{r})$ with

$$\begin{aligned} &\hat{A}_i = \hat{T}_2 A_i R_2, \ \hat{B}_i = \hat{T}_2 B_i, \ \hat{C} = C\hat{T}_2, \\ &\hat{r} = r - \tilde{r} < r. \end{aligned} \tag{64}$$

■

Remark 4: The so-called CE approach in [24] states that if $A_1, \ldots, A_n$ share a CE $\boldsymbol{w}$ such that (36) holds true. As clarified in Subsection III-B, the reducibility condition given in CE approach [24] always requires the existence of CCIS. However, it is not right to say the opposite. As a result, the reducibility conditions in [20, 24] can only be viewed as the eigenvector-based IS of dimension one. Therefore, the CCIS approach is more general and practical than the existing methods.

C. Necessary reducibility conditions for F-M models

This subsection shows that the sufficient reducibility condition given in the previous subsection is also necessary in the non-commutative setting (NCS). To this end, the following definitions are required.

Definition 9: A formal power series of the TFM in the NCS is defined by

$$H(z_1, \ldots, z_n) = \sum_{w \in \mathcal{F}_{\mathcal{N}}} H_w z^w, \tag{65}$$

where H_w is the coefficient matrix w.r.t.

$$z^w = z_{\alpha_l} z_{\alpha_{l-1}} \cdots z_{\alpha_1}, \tag{66}$$

with $w = \alpha_1 \alpha_2 \cdots \alpha_l$. Note that here the variables $z_1, \ldots, z_n$ in (65) are not commutative, i.e.,

$$z_{\alpha_1} z_{\alpha_2} \neq z_{\alpha_2} z_{\alpha_1}$$

for all $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in \boldsymbol{n}$.

Now, the TFM of an F-M model in the NCS can be stated as follows.

Definition 10: For the TFM $H(z_1, \ldots, z_n)$ of (43), if one assumes the unite delay operators $z_1, \ldots, z_n$ are non-commutative, i.e., $z_{\alpha_1} z_{\alpha_2} \neq z_{\alpha_2} z_{\alpha_1}$ for $\alpha_1 \neq \alpha_2$ and $\alpha_1, \alpha_2 \in \mathcal{N}$, then such a TFM is called non-commutative. In other words, the non-commutative TFM of the F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ is expressed as

$$\begin{aligned} H(z_1, \ldots, z_n) =& C \sum_{k=0}^{\infty} \left(\sum_{i=1}^{n} A_i z_i \right)^k \left(\sum_{i=1}^{n} B_i z_i \right) + D \\ =& \sum_{w \in \mathcal{F}_{\mathcal{N}}^+} C A_v B_{\alpha_1} z^w + D, \end{aligned} \tag{67}$$

where $w = \alpha_1 \alpha_2 \cdots \alpha_l = \alpha_1 v \in \mathcal{F}_{\boldsymbol{n}}^+$, A_v and z^w are defined in (24) and (66), respectively.

Next, the realization and reduction of the F-M model in the NCS can be stated as follows.

Definition 11: For a given TFM $H(z_1, \ldots, z_n)$ in the form of (65) if there exist matrices $A_1, \ldots, A_n$, $B_1, \ldots, B_n$, C and D such that

$$C A_v B_{\alpha_1} = H_w, \ D = H(\epsilon), \tag{68}$$

where $w = \alpha_1 \alpha_2 \cdots \alpha_l = \alpha_1 v \in \mathcal{F}_{\boldsymbol{n}}^+$, A_v and z^w are defined in (24) and (66), respectively, then $(\mathcal{A}, \mathcal{B}, C, D; r)$ is called an F-M model realization of (65) in the NCS.

Definition 12: For a given F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$, if there is a new one $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \mathcal{C}, D; \hat{r})$ satisfying

$$\hat{C} \hat{A}_v \hat{B}_{\alpha_1} z^w = C A_v B_{\alpha_1} z^w, \ \hat{r} \leq r, \tag{69}$$

for all with $w = \alpha_1 \alpha_2 \cdots \alpha_l = \alpha_1 v \in \mathcal{F}_{\mathcal{N}}^+$, A_v and z^w being defined in (24) and (66), respectively, then the given F-M model is reducible.

The necessary reducibility condition for the F-M model in the NCS can be given as follows.

Theorem 8: For an F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$, if there is another one $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, \hat{D}; \hat{r})$ with $\hat{r} < r$ such that they share the same non-commutative TFM, then the F-M model $(\mathcal{A}, \mathcal{B}, C, D; r)$ admits a left or right CCIS.

Proof: Assume that there is a reduced-order F-M model $(\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{C}, \hat{D}; \hat{r})$ for the one $(\mathcal{A}, \mathcal{B}, C, D, r)$ with $\hat{r} < r$, but $(\mathcal{A}, \mathcal{B}, C, D; r)$ has no left and right CCIS. We derive

from the same non-commutative TFMs of $(\mathcal{A},\mathcal{B},C,D;r)$ and $(\hat{\mathcal{A}},\hat{\mathcal{B}},\hat{C},\hat{D};\hat{r})$ that

$$\begin{aligned}&\mathcal{H}_r(\mathcal{A},B,C)=\mathcal{H}_r(\hat{\mathcal{A}},\hat{B},\hat{C})\\=&\mathcal{M}_r(\hat{\mathcal{A}},\hat{C})\mathcal{M}_r^\top(\hat{\mathcal{A}}^\top,B^\top).\end{aligned}\tag{70}$$

Note that $\mathcal{M}_r(\hat{\mathcal{A}},\hat{C})\in\mathbb{R}^{\frac{p(1-n^{\hat{r}})}{1-n}\times\hat{r}}$, we have

$$\operatorname{rank}\left(\mathcal{M}_r(\hat{\mathcal{A}},\hat{C})\right)\le\hat{r}<r.\tag{71}$$

According to Sylvester inequality [23], we obtain

$$\operatorname{rank}\big(\mathcal{H}_r(\mathcal{A},B,C)\big)=\operatorname{rank}\big(\mathcal{H}_r(\hat{\mathcal{A}},\hat{B},\hat{C})\big)<r.\tag{72}$$

By Theorem 4, we have that the F-M model $(\mathcal{A},\mathcal{B},C,D;r)$ has a left or right CCIS, which contradicts the assumption. The proof is completed. ■

V. State-feedback control of rational parameter systems

This section proposes an H_∞ gain-scheduled state-feedback controller synthesis for rational parameter systems to further show the superiority of the established CCIS.

Consider the discrete-time rational parameter system (DTRPS) [33]

$$x_{k+1}=A(\theta_k)x_k+B(\theta_k)u_k+B_w(\theta_k)w_k,\tag{73a}$$
$$z_k=C_z(\theta_k)x_k+D_{zu}(\theta_k)u_k+D_{zw}(\theta_k)w_k.\tag{73b}$$

where $x_k\in\mathbb{R}^r$, $u_k\in\mathbb{R}^q$, $w_k\in\mathbb{R}^n$, $z_k\in\mathbb{R}^m$ are the system state, the control input, the disturbances, and the controlled output, respectively; and all the matrices are rational and continuous in time-varying parameters collected in $\theta_k\in\Theta=\{(\theta_{k1},\dots,\theta_{kn})\in\Theta_1\times\cdots\times\Theta_n\}$. Here, Θ_i is the convex hull of the vertices $\theta_i^{[1]},\dots,\theta_i^{[v_i]}$, i.e., $\theta_{ki}\in\Theta_i$ can be expressed as $\theta_{ki}=\sum_{l_i=1}^{v_i}x_{l_i}\theta_i^{[l_i]}$ with $\sum_{l_i=1}^{v_i}x_{l_i}=1$ and $x_{l_i}\ge 0$, $l_i=1,\dots,v_i$. We denote by $\mathcal{L}\triangleq\mathcal{L}_1\times\cdots\times\mathcal{L}_n$ the set of all extremal values of the parameters in θ_k and an element of $\mathcal{L}$ will be denoted by $\theta^{[l]}$ with $l=(l_1,\dots,l_n)$. Then, it can be found that the total number of vertices in Θ is $N_\theta=\prod_{i=1}^n v_i$.

Assuming that the parameter θ_k is available online for controller implementation, we are interested in designing gain-scheduled state-feedback controllers of the form

$$u_k=K(\theta_k)x_k,\quad K:\Theta\to\mathbb{R}^{q\times r}\tag{74}$$

such that

$$\mu_d=\sup_{\|w_k\|_2\neq 0}\frac{\|z_k\|_2}{\|w_k\|_2}.\tag{75}$$

Algebraically, let $\theta_{ki}=z_i$, $i=1,\dots,n$. It is noted that

$$H(z)\triangleq H(\theta_k)=\begin{bmatrix}A^\top(\theta_k) & C_z^\top(\theta_k)\\ B^\top(\theta_k) & D_{zu}^\top(\theta_k)\\ B_w^\top(\theta_k) & D_{zw}^\top(\theta_k)\end{bmatrix}\tag{76}$$

can be represented in the F-M model, i.e.,

$$\begin{aligned}H(\theta_k)=&C\left(I_r-\sum_{i=1}^n\theta_{ki}A_i\right)^{-1}\left(\sum_{i=1}^n\theta_{ki}B_i\right)+D\\=&\begin{bmatrix}D & C\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & I_r-\sum_{i=1}^n\theta_{ki}A_i\end{bmatrix}^{-1}\begin{bmatrix}I\\ \sum_{i=1}^n\theta_{ki}B_i\end{bmatrix}\\=&\begin{bmatrix}E_{dx}\\ E_{dy}\\ E_{dz}\end{bmatrix}E_{d\pi}^{-1}(\theta_k)M_{3d}(\theta_k),\end{aligned}\tag{77}$$

with $M_{3d}(\theta_k)=\begin{bmatrix}A_d(\theta_k) & B_{dw}(\theta_k)\end{bmatrix}$.

Define

$$\eta_{dk}^\top\triangleq\begin{bmatrix}x_{d(k+1)}^\top & z_{dk}^\top & \pi_{dk}^\top & x_{dk}^\top & w_{dk}^\top\end{bmatrix},$$
$$E_d(\theta_k)\triangleq\begin{bmatrix}I & 0 & E_{dx}+K^\top(\theta_k)E_{dy} & 0 & 0\\ 0 & I & E_{dz} & 0 & 0\\ 0 & 0 & E_{d\pi}(\theta_k) & A_d(\theta_k) & B_{dw}(\theta_k)\end{bmatrix}.$$

It can be verified that

$$E_d(\theta_k)\eta_{dk}=0.\tag{78}$$

Then, the H_∞ performance of (73) in closed-loop is equal to that of $E_d(\theta)\eta_{dk}=0$. The specific process is omitted, which is similar to Lemma 2 in [33].

Define

$$N_{dx}(\theta_k)=\begin{bmatrix}S_{dx}(\theta_k) & 0 & S_{dx}(\theta_k)E_{dx}+S_{dy}(\theta_k)E_{dy} & 0 & 0\end{bmatrix},$$
$$N_{d\pi}(\theta_k)=\begin{bmatrix}0 & I & E_{dz} & 0 & 0\\ 0 & 0 & E_{d\pi}(\theta_k) & A_d(\theta_k) & B_{dw}(\theta_k)\end{bmatrix}.$$

Then, the gain-scheduled state-feedback controller for the DTRPS can be given by the following result.

Theorem 9: If there exist N_θ symmetric positive definite matrices $P_d^{[l]},P_d^{[l^+]}$, $S_{dx}^{[l]},S_{dy}^{[l]}$ and matrix $S_{d\pi}$ of appropriate size such that

$$\begin{aligned}&\operatorname{diag}\left(P_d^{[l^+]},\quad I,\quad 0,\quad -P_d^{[l]},\quad -\mu_d^2I\right)\\ \prec&\left\{\begin{bmatrix}I\\0\end{bmatrix}N_{dx}\left(\theta_k^{[l]}\right)\right\}^s+\left\{S_{d\pi}N_{d\pi}\left(\theta_k^{[l]}\right)\right\}^s,\end{aligned}\tag{79}$$

hold simultaneously, then the gain-scheduled state-feedback gains are given by $K^{[l]}=(S_{dy}^{[l]})^\top\left((S_{dx}^{[l]})^\top\right)^{-1}$.

Proof: First, let $P_d^{[l]}$ denote the vertices of $P_d(\theta_k)$, and then $P_d^{[l^+]}$ denote the vertices of $P_d(\theta_{k+1})$. By convexity of matrix inequalities, we derive that the LMIs of (79) hold if and only if

$$\begin{aligned}&\operatorname{diag}\left(P_d(\theta_{k+1}),\quad I,\quad 0,\quad -P_d(\theta_k),\quad -\mu_d^2I\right)\\ \prec&\left\{\begin{bmatrix}I\\0\end{bmatrix}N_{dx}(\theta_k)\right\}^s+\left\{S_{d\pi}N_{d\pi}(\theta_k)\right\}^s,\end{aligned}\tag{80}$$

for all uncertainties $\theta\in\Theta$. With the change of variable $S_{dy}(\theta_k)=S_{dx}(\theta_k)K^\top(\theta_k)$, we have

$$\begin{aligned}&\operatorname{diag}\left(P_d(\theta_{k+1}),\quad I,\quad 0,\quad -P_d(\theta_k),\quad -\mu_d^2I\right)\\ &\prec\left\{S_d(\theta_k)E_d(\theta_k)\right\}^s,\end{aligned}\tag{81}$$

with $S_d(\theta_k) = \left[\begin{array}{c|c} S_{dx}(\theta_k) & \\ 0 & S_{d\pi} \end{array}\right]$. After congruence operation of $\eta_{dk} \neq 0$ on this last matrix inequality, one gets along the trajectories ($E_d(\theta_k)\eta_{dk} = 0$) that

$$\begin{aligned} & x_{d(k+1)}^\top P_d(\theta_{k+1}) x_{d(k+1)} - x_{dk}^\top P_d(\theta_k) x_{dk} \\ & + z_{dk}^\top z_{dk} - \mu_d^2 w_{dk}^\top w_{dk} < 0, \end{aligned} \tag{82}$$

which is enough to conclude about asymptotic stability (strictly decreasing Lyapunov function $x_d^\top P_d(\theta_k) x_d$ for zero disturbances) and induced norm performance $\|z_d\|_2 \leq \mu_d \|w_d\|_2$ for zero initial conditions. ■

Remark 5: It can be observed that the F-M in (77) plays an essential role in designing a gain-scheduled state-feedback controller for a rational parameter system in the form of (73). In order to reduce the numerical complexity and computation in the LMIs of (79), a lower-order F-M model is desired by applying the established CCIS approach.

Example 4: In order to demonstrate the effectiveness of the proposed gain-scheduled state-feedback controller and the CCIS approach, consider the DTRPS of (73) with

$$\begin{aligned} & A(\theta) = \begin{bmatrix} -\theta_2(1+\theta_2)^{-1} & -\theta_2(1+\theta_2)^{-1} \\ 1 & (\theta_1+\theta_1\theta_2+\theta_3)(1+\theta_1+\theta_2)^{-1} \end{bmatrix}, \\ & B(\theta) = \begin{bmatrix} 0 \\ \theta_2 \end{bmatrix}, \ B_w(\theta) = \begin{bmatrix} (1+\theta_2)^{-1} \\ 0 \end{bmatrix}, \ D_{zw}(\theta) = 1, \\ & C_z(\theta) = \begin{bmatrix} 1 & -\theta_1\theta_2 \end{bmatrix}, \ D_{zu}(\theta) = \theta_1 + \theta_2. \end{aligned} \tag{83}$$

The three parameters $\theta_1, \theta_2, \theta_3$ are in intervals around the nominal value 0 with discrepancies $\delta_1, \delta_2, \delta_3$, i.e., $\theta_i \in [-\delta_i, \delta_i]$, $i = 1, 2, 3$.

In order to conduct a state feedback design for the DTRPS of (83), we need to obtain a 3-D F-M model for $H(\theta_k)$ in the form of (76). Applying the F-M model realization method of [8] to $H(\theta_k)$ yields an F-M model with order 10:

$$\begin{aligned} & A_1 = \begin{bmatrix} \mathbf{0} & \boldsymbol{e}_2 & -\boldsymbol{e}_3 & -\boldsymbol{e}_3+\boldsymbol{e}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{e}_{10} & \mathbf{0} \end{bmatrix}, \\ & A_2 = \begin{bmatrix} -\boldsymbol{e}_1 & \mathbf{0} & -\boldsymbol{e}_4 & -\boldsymbol{e}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ & A_3 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{e}_5 & -\boldsymbol{e}_5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ & B_1 = \begin{bmatrix} \mathbf{0} & \boldsymbol{e}_3 & \boldsymbol{e}_8 \end{bmatrix}, \ B_2 = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_5 & \boldsymbol{e}_9 \end{bmatrix}, \\ & B_3 = \begin{bmatrix} \mathbf{0} & \boldsymbol{e}_4 & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \\ & C = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{84}$$

where $\boldsymbol{e}_k \in \mathbb{R}^{10}$ represents a column vector with entries being zeros except the k-th entry being 1, $k \in \{1, \ldots, 10\}$.

In order to reduce the complexity of the analysis, we can apply the established CCIS approach to get a much lower-order F-M model:

$$\begin{aligned} & \hat{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \\ & \hat{A}_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ & \hat{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \hat{B}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ & \hat{C} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \hat{D} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{85}$$

whose order is 6. Taking $\delta_1 = \delta_2 = \delta_3 = 0.25$, and applying Theorem 9 to the different order F-M model of (84) and (85) yields respectively state feedback gains

$$\begin{aligned} & K_1 = \begin{bmatrix} 3.5228 & -0.7570 \end{bmatrix}, \ K_2 = \begin{bmatrix} 3.7938 & -0.1119 \end{bmatrix}, \\ & K_3 = \begin{bmatrix} -1.4093 & 0.2767 \end{bmatrix}, \ K_4 = \begin{bmatrix} -1.1868 & 0.1637 \end{bmatrix}, \\ & K_5 = \begin{bmatrix} 1.8395 & 0.1256 \end{bmatrix}, \ K_6 = \begin{bmatrix} 2.1208 & 0.8443 \end{bmatrix}, \\ & K_7 = \begin{bmatrix} -2.6891 & 0.1937 \end{bmatrix}, \ K_8 = \begin{bmatrix} -2.4460 & 0.0371 \end{bmatrix}; \end{aligned} \tag{86}$$

and

$$\begin{aligned} & \hat{K}_1 = \begin{bmatrix} 3.5285 & -0.7321 \end{bmatrix}, \ \hat{K}_2 = \begin{bmatrix} 3.7273 & -0.0569 \end{bmatrix}, \\ & \hat{K}_3 = \begin{bmatrix} -1.2306 & 0.3414 \end{bmatrix}, \ \hat{K}_4 = \begin{bmatrix} -0.8984 & 0.1042 \end{bmatrix}, \\ & \hat{K}_5 = \begin{bmatrix} 1.8181 & 0.2190 \end{bmatrix}, \ \hat{K}_6 = \begin{bmatrix} 2.0795 & 0.8474 \end{bmatrix}, \\ & \hat{K}_7 = \begin{bmatrix} -2.5943 & 0.2026 \end{bmatrix}, \ \hat{K}_8 = \begin{bmatrix} -2.3460 & 0.1075 \end{bmatrix}. \end{aligned} \tag{87}$$

Let $\delta_1 = \delta_2 = \delta_3 = \delta$, take different δ, and carry out the state feedback design. The comparisons are given by Table I. It is noted that Table I depicts the total rows of all LMI restrictions N_r, the number of necessary scalar decision variables N_v, and the size of the Lyapunov matrices N_p, which are required to solve the problems and directly related to the overall computational complexity. It shows that lower-order models greatly reduce numerical complexity and computation and may also improve state-feedback control of the rational parameter systems.

VI. Conclusions

The notion of CCIS has been proposed to take into account the overall structural characteristics of all state matrices. The necessary and sufficient conditions for the existence of CCIS have been established, and the corresponding calculation algorithm has been given. It has been proved that the existence

TABLE I
NUMERICAL RESULTS OF STATE FEEDBACK FOR F-M MODELS WITH DIFFERENT ORDERS

δ	μ_c		(N_r, N_v)		N_p	
	10-order	6-order	10-order	6-order	10-order	6-order
0.250	5.9332	5.9325				
0.256	7.3516	7.3494				
0.262	9.8377	9.8417	$(339, 1218)$	$(223, 962)$	$(19, 14)$	$(15, 10)$
0.267	13.9263	13.9233				
0.272	24.928	24.9144				

of a CE leads to the existence of CCIS, but not vice versa, so the established CCIS can better reveal the internal structural properties of multiple matrices than the CE. Based on this CCIS, the necessary and sufficient reducibility conditions and the associated reduction algorithms for the F-M model have been developed. Moreover, a gain-scheduled state-feedback control has been proposed for rational parameter systems, the numerical complexity of which can be reduced greatly by applying CCIS. Examples have been provided to highlight the details and effectiveness of the new approach.

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